JOURNAL OF APPROXIMATION THEORY **34**, 379–383 (1982)

Polynomial Approximation with Bounds

LAWRENCE ZALCMAN

Department of Mathematics, University of Maryland, College Park, Maryland 20742

Communicated by Oved Shisha

Received March 8, 1981

Let Γ be a proper closed arc of the unit circle T. According to the Weierstrass Approximation Theorem, each function continuous on Γ can be approximated uniformly by polynomials in z $(=e^{i\theta})$. If we require additionally that the suprema of the approximating polynomials remain uniformly bounded on the open unit disc Δ, the possibility of approximation becomes severely limited.

THEOREM 1. *A function $f \in C(\Gamma)$ is uniformly approximable (on Γ) by polynomials p_n satisfying $|p_n(z)| \leqslant M$, $z \in \Delta$, if and only if there exists a function g analytic on Δ, $|g(z)| \leqslant M$, such that*

$$f(e^{i\theta}) = \lim_{r \to 1} g(re^{i\theta}), \qquad e^{i\theta} \in \Gamma.$$

Remark. It follows easily from the Poisson representation that the function g actually extends continuously to $\Delta \cup \gamma$, where γ is the open arc obtained by deleting the endpoints of Γ; at these endpoints, g may actually have a nontrivial cluster set (though, of course, its extension has the appropriate one-sided limits there).

Proof. Suppose first that f is so approximable. Let $\Delta_r = \{z : |z| \leqslant r\}$ and let D_r be the closed convex hull of $\Delta_r \cup \Gamma$. We claim that the polynomials p_n converge uniformly on D_r for each r, $0 < r < 1$. For this it is sufficient to show that the sequence $\{p_n\}$ is uniformly Cauchy on each D_r.

Denote the Poisson kernel for $z \in \Delta$ by $P_z(\theta)$. Since $\log|p_n - p_m|$ is subharmonic, we have

$$\log|p_n(z) - p_m(z)| \leqslant \int_0^{2\pi} \log|p_n(e^{i\theta}) - p_m(e^{i\theta})| \, P_z(\theta)\, d\theta$$

$$= \int_{T \setminus \Gamma} + \int_{\Gamma}.$$

0021-9045/82/040379-05$02.00/0

Copyright © 1982 by Academic Press, Inc.
All rights of reproduction in any form reserved.

The first integral on the right is clearly bounded by $\log 2M$. For n and m large enough, the second integral is bounded by $\omega_\Gamma(z)\log\varepsilon_{nm}$, where ω_Γ is the harmonic measure of Γ and

$$\varepsilon_{nm} = \max_\Gamma |p_n(e^{i\theta}) - p_m(e^{i\theta})| < 1.$$

Letting $C_r = \min_{D_r} \omega_\Gamma(z) > 0$, we have

$$\sup_{D_r} \log |p_n(z) - p_m(z)| \leqslant \log 2M + C_r \log \varepsilon_{nm},$$

which tends to $-\infty$ as $n, m \to \infty$. Thus

$$\lim_{n,m\to\infty} \sup_{D_r} |p_n(z) - p_m(z)| = 0.$$

as required.

Conversely, suppose that g satisfies the conditions of the theorem. Then g is continuous, hence uniformly continuous, on the closed set $\{re^{i\theta}: e^{i\theta} \in \Gamma, 0 \leqslant r \leqslant 1\}$. Thus, for r sufficiently close to 1, the function $g_r(z) = g(rz)$ approximates $f(z)$ closely for $z = e^{i\theta} \in \Gamma$. Since g_r is analytic on the closed unit disc and bounded by M it can be approximated uniformly on the unit circle by a polynomial with norm no greater than M. This polynomial approximates f on Γ.

The first half of the argument given above actually shows that if the p_n approximate f uniformly on Γ and

$$\int_0^{2\pi} \log^+ |p_n(e^{i\theta})|\, d\theta \leqslant M, \qquad n = 1, 2, \ldots$$

then f has an analytic continuation into the full unit disc. It would be interesting to determine how badly unbounded the polynomial approximants of a function which does *not* admit such continuation must be.

One can ring the changes on Theorem 1 by altering variously the set on which one approximates, the sense in which approximation is required to hold, and the precise conditions of boundedness. A typical example is provided by the following result.

THEOREM 2. *Let $E \subset T$ be a set of positive measure and let $q \geqslant 1$. A function f on E is the (pointwise, almost everywhere) limit of polynomials p_n satisfying $\|p_n\|_q \leqslant M$ if and only if there exists a function $g \in H^q$, $\|g\|_q \leqslant M$, such that*

$$f(e^{i\theta}) = g(e^{i\theta}) \equiv \lim_{r\to 1} g(re^{i\theta}) \qquad \text{a.a.} \quad e^{i\theta} \in E.$$

Proof. Since $\|p_n\|_q \leqslant M$, the functions p_n are uniformly bounded on each compact subset of Δ and hence form a normal family. Thus, a subsequence, which we again denote by $\{p_n\}$, converges uniformly on compacta to a function g analytic on Δ. Since for each $0 < r < 1$

$$\frac{1}{2\pi}\int_0^{2\pi} |g(re^{i\theta})|^q \, d\theta = \lim_{n\to\infty} \frac{1}{2\pi}\int_0^{2\pi} |p_n(re^{i\theta})|^q \, d\theta$$
$$\leqslant \overline{\lim_{n\to\infty}} \frac{1}{2\pi}\int_0^{2\pi} |p_n(e^{i\theta})|^q \, d\theta \leqslant M^q,$$

it is clear that $g \in H^q$, $\|g\|_q \leqslant M$. Now by Hölder's inequality, $\|p_n\|_1 \leqslant \|p_n\|_q \leqslant M$, so for some subsequence, which we again denote by $\{p_n\}$, the measures $p_n(e^{i\theta})\,d\theta$ tend weak $*$ to a measure μ which satisfies

$$\int P_z \, d\mu = \lim_{n\to\infty} \int P_z p_n \, d\theta = \lim_{n\to\infty} p_n(z)$$
$$= g(z) = \int P_z \, g \, d\theta$$

for each $z \in \Delta$. By the uniqueness theorem for the Poisson integral, $d\mu = g(e^{i\theta})\,d\theta$.

Now since $p_n \to f$ a.e. on E, Fatou's lemma yields

$$\int_E |f| \, d\theta \leqslant \underline{\lim_{n\to\infty}} \int_T |p_n| \, d\theta \leqslant 2\pi M,$$

so that f is finite a.e. on E. By Egoroff's theorem, there exists a sequence of sets $E_1 \subset E_2 \subset \cdots$ contained in E such that $p_n \to f$ uniformly on each E_k and $E \backslash \bigcup E_k$ has measure zero. Fixing k, we have

$$\int P_z \, g \, d\theta = \lim_{n\to\infty} \int P_z p_n \, d\theta$$
$$= \lim_{n\to\infty} \left\{ \int_{E_k} P_z p_n \, d\theta + \int_{T\backslash E_k} P_z p_n \, d\theta \right\}$$
$$= \int_{E_k} P_z f \, d\theta + \lim_{n\to\infty} \int_{T\backslash E_k} P_z p_n \, d\theta$$

for all $z \in \Delta$. The uniqueness theorem shows that $g(e^{i\theta}) = f(e^{i\theta})$ a.e. on E_k and hence a.e. on E.

Conversely, suppose $g \in H^q$. The functions $g_n(z) = g((1 - 1/n)\,z)$ are analytic on the closed disc and satisfy $\|g_n\|_q \leqslant \|g\|_q$. Approximate each g_n

uniformly on the closed disc to within $1/n$ by a polynomial p_n satisfying $\|p_n\|_q \leqslant \|g_n\|_q$. Since $g_n(e^{i\theta}) \to g(e^{i\theta})$ a.e. on T, we have also $p_n(e^{i\theta}) \to g(e^{i\theta})$ a.e.

The argument given above actually shows that the full sequence $\{p_n\}$ converges to g uniformly on compact subsets of Δ. Indeed, each subsequence of $\{p_n\}$ contains a convergent subsequence the boundary values of whose limit agree with f a.e. on E, a set of positive measure. Thus any two such limits must be identical. Actually, the first half of the proof of Theorem 1 can be adapted to give a proof of the corresponding part of Theorem 2; and, conversely, it is evident that the argument used in Theorem 2 applies equally well to Theorem 1.

In case the boundedness hypothesis is strengthened to require that the l^1 norms of the approximating polynomials remain uniformly bounded, the possibility of nontrivial approximation evaporates completely: the only functions so approximable are (restrictions of) absolutely convergent Taylor series. While this follows fairly routinely from some general functional analysis, it is just as easy to give a direct proof.

Indeed, let $p_n(z) = \sum_k a_k(n) z^k$ and suppose that $\sum_k |a_k(n)| \leqslant M$ for $n = 1, 2, \ldots$. Since $p_n(z)$ converges uniformly to $g(z) = \sum a_k z^k$ on a neighborhood of 0, we have, for each k, $a_k(n) \to a_k$ as $n \to \infty$. We claim that $\sum_k |a_k| \leqslant M$. Otherwise, there exists N such that $\sum_{k=0}^{N} |a_k| > M$. Choosing unimodular constants c_k $(0 \leqslant k \leqslant N)$ so that $a_k c_k = |a_k|$, we have

$$M \geqslant \sum_{k=0}^{N} |a_k(n)| \geqslant \left| \sum_{k=0}^{N} a_k(n)\, c_k \right| \to \sum_{k=0}^{N} a_k c_k$$
$$= \sum_{k=0}^{N} |a_k| > M,$$

a contradiction.

The knowledgeable reader will recognize the close connection between the results discussed above and the Khintchine–Ostrowski theorem: a uniformly bounded sequence of functions analytic on Δ which converges on a subset E of T having positive measure converges uniformly on compact subsets of Δ; cf. [1–4].

Acknowledgments

I should like to thank Shaul Foguel for having asked me the question which prompted this note. This paper was prepared while the author was a visiting professor at the Hebrew University of Jerusalem and Bar-Ilan University. Research supported by NSF Grant MCS 78–00811 A02, the General Research Board of the University of Maryland, and the Lady Davis Fellowship Trust.

References

1. A. Khintchine, Sur les suites de fonctions analytiques bornées dans leur ensemble, *Fund. Math.* **4** (1923), 72–75.
2. A. Khintchine, On sequences of analytic functions, *Recueil Math. Moscou* (*Mat. Sb.*) **31** (1922–24), 147–151. [Russian]
3. A. Ostrowski, Auszug aus einem Briefe von A. Ostrowski an L. Bieberbach, *Jber. Deutsch. Math.-Verein.* **31** (1922), 82–85.
4. A. Ostrowski, Über die Bedeutung der Jensenschen Formel für einige Fragen der komplexen Funktionentheorie, *Acta Lit. Sci. Regiae Univ. Hungar. Francisco-Josephinae* **1** (1922–23), 80–87.